Groundwater hydraulics
Water flow
in porous media,
hydraulic conductivity,
Darcy's law

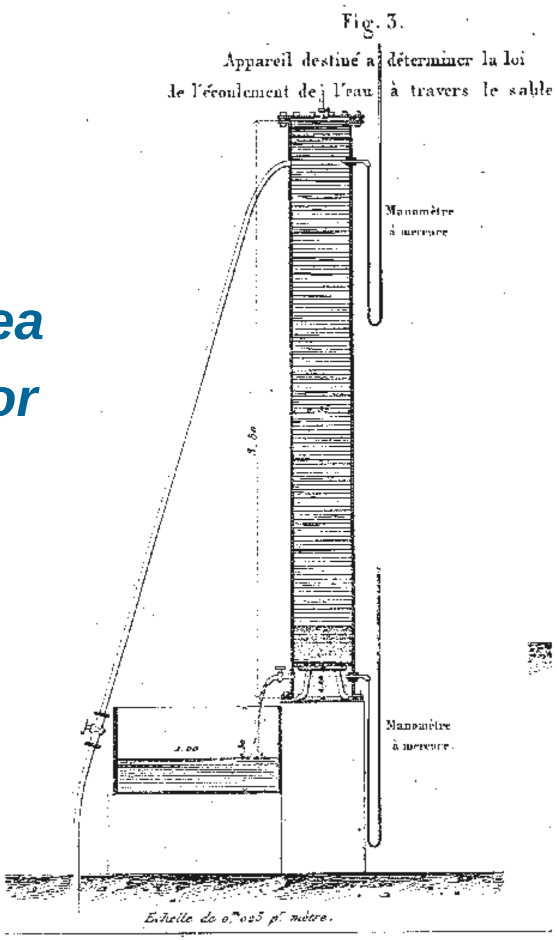
Saturated flow

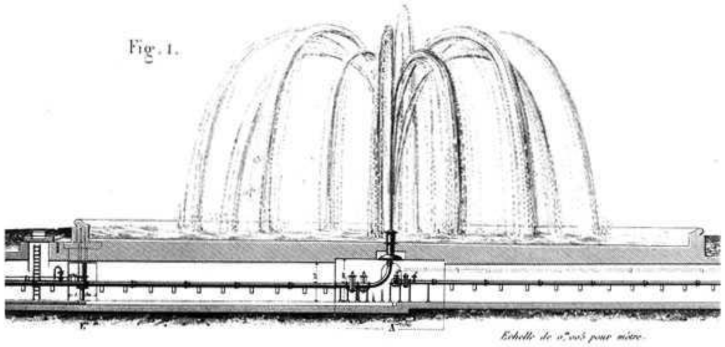
Henri Darcy (1856) *filtration of water for fountains in Dijon*

After many experiments he found that ***water flow*** *through the soil column depends on:*

- ***directly proportional to pressure drop***
- ***inversely proportional to the length***
- ***directly proportional to the crossectional area***
- ***dependent on coefficient which is specific for each media***

Henry Darcy

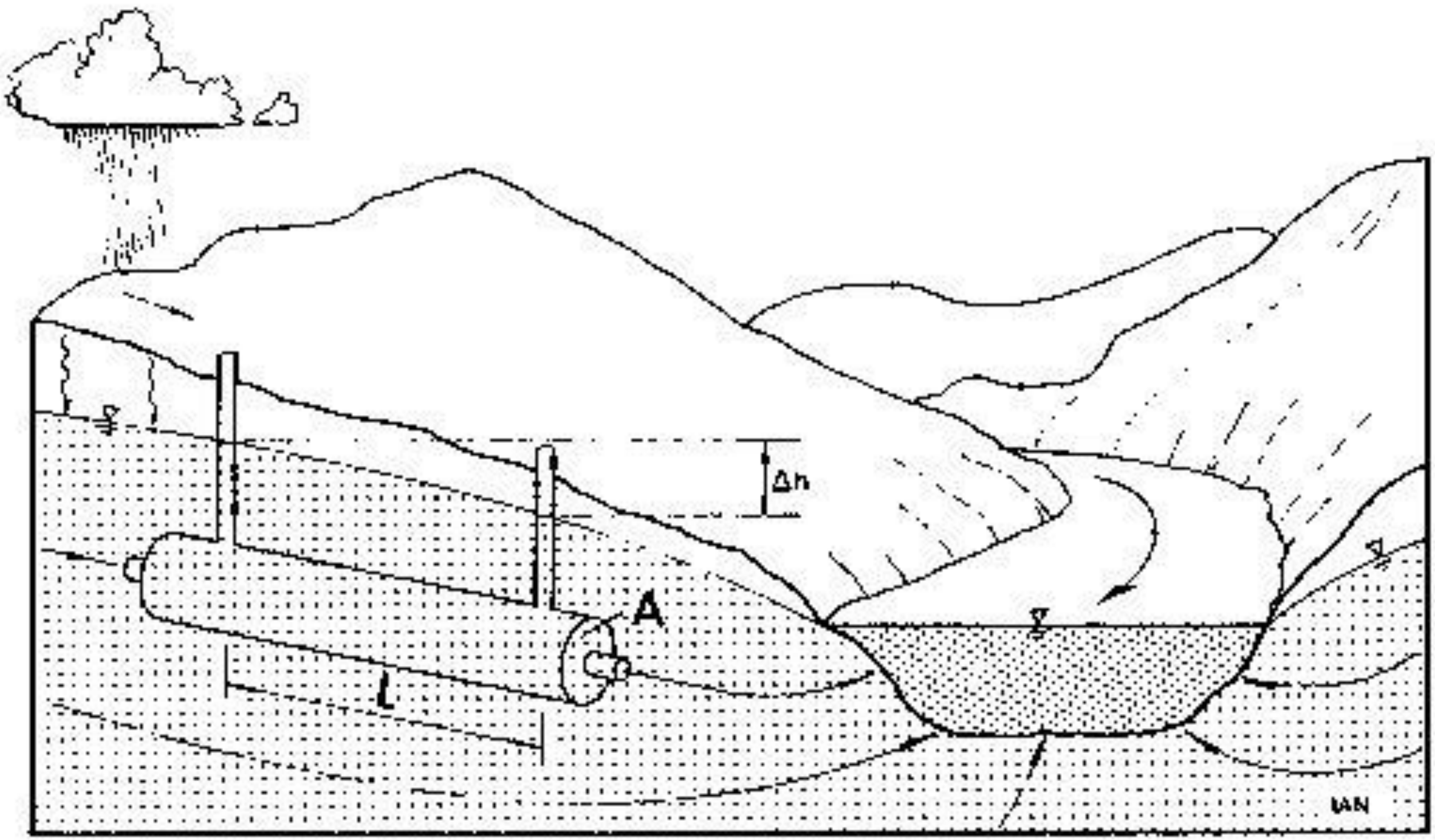




Darcy, H., 1856. *Les Fountaines de la Ville de Dijon*

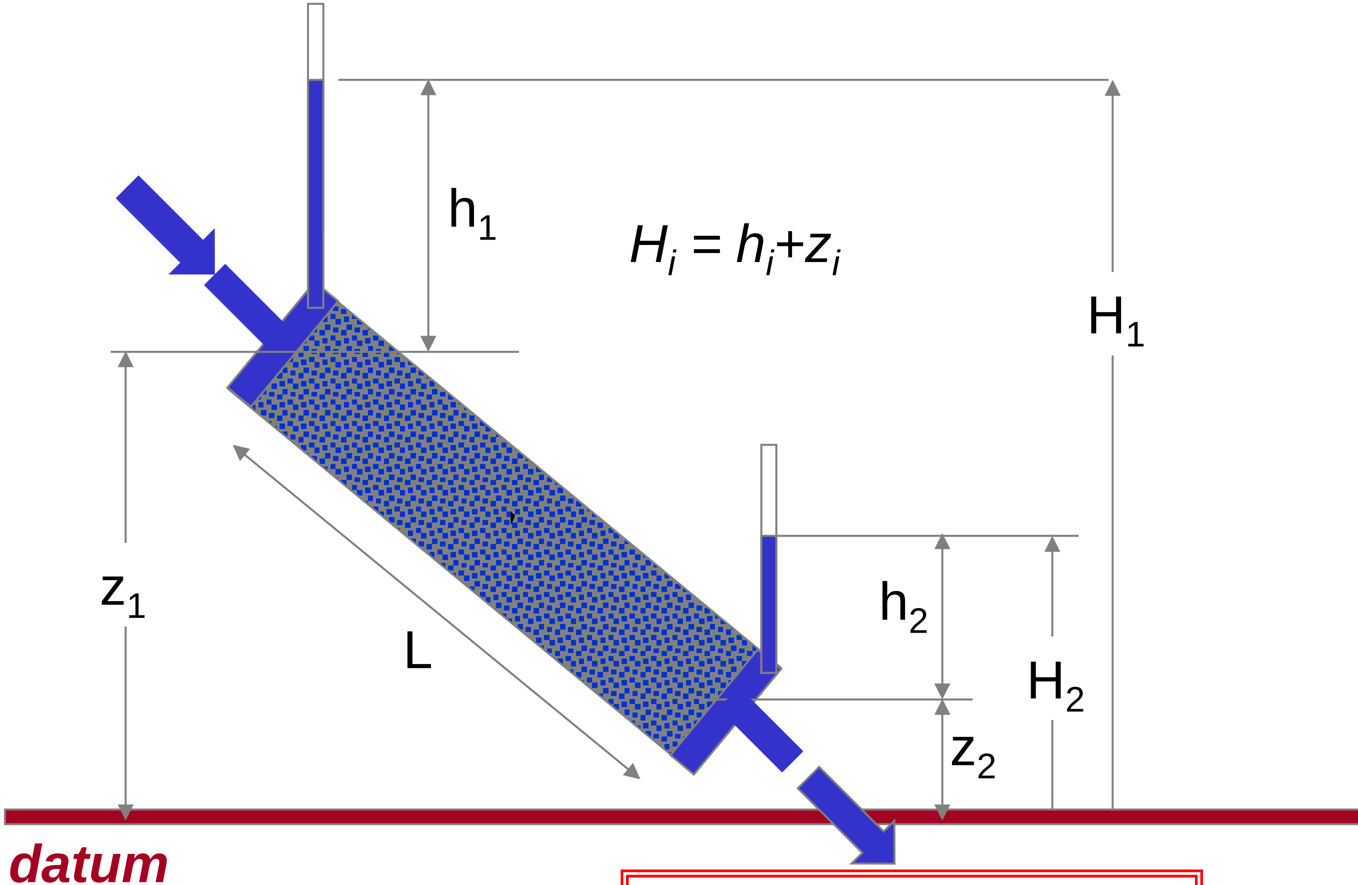
Darcy's law

Henri Darcy 1856



Darcy law

$$Q = \frac{K_S A \Delta H}{L}$$



valid in fully saturated porous media
For example: under the ground water level

Q = flow [$L^3.T^{-1}$]
A = crossectional area [L^2]
K_s = saturated hydraulic conductivity [$L.T^{-1}$]
$\Delta H = H_1 - H_2$ (hydraulic head drop) [L]
L= sample lenght [L]

for:

$$q = \frac{Q}{A}$$

kde:
q ... Volume flux [L.T^-1]
Q ... Flow rate [L^3.T^-1]
A ... Crossectional area [L^2]

Transforms to the:

$$q = K_s \frac{\Delta H}{L}$$

note**: **negative sign** due to the fact **grad H** aims **against flow direction

More gereneral form:

$$q = K_s \frac{dH}{dl}$$

Pro 1D vertical flow

$$q = -K_s \frac{dH}{dl} = -K_S \nabla H$$

For 3D flow

$$v_x = -K \frac{\partial H}{\partial x}$$

$$v_y = -K \frac{\partial H}{\partial y}$$

$$v_z = -K \frac{\partial H}{\partial z}$$

Coefficient or the saturated hydraulic conductivity K_s

Also called (sometimes) ***filtration coefficient**, **darcy's coefficient*** *or* ***permeability (incorrect)***

commonly used units K_s ($m.s^{-1}$), ($cm.d^{-1}$), ($cm.s^{-1}$)

K_s is property of water-solid interaction.

Parameter which is related only to the porous media (independently on flowing liquid)
is:

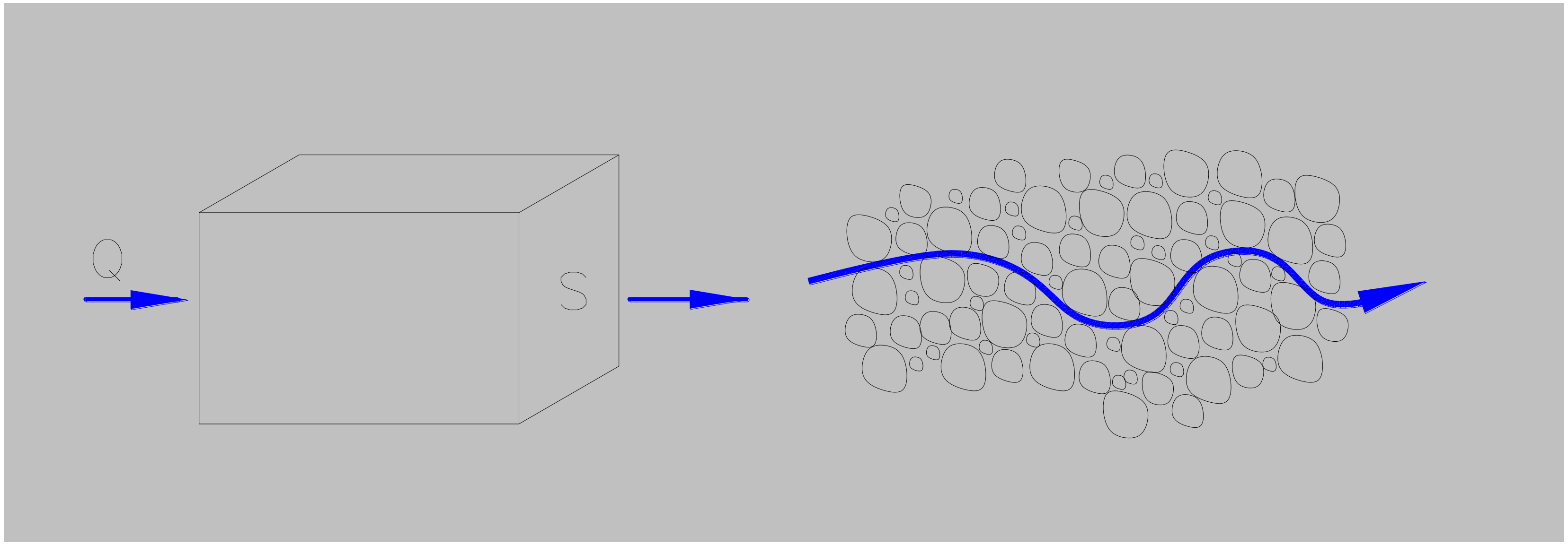
Permeability k

$k = K_s \nu / g$ [L^2]

$$K = \frac{k \, \rho \, g}{\mu}$$

where ν kinematic viscosity

when μ is dynamic viscosity



Darcian velocity

$$v = \frac{Q}{S}$$

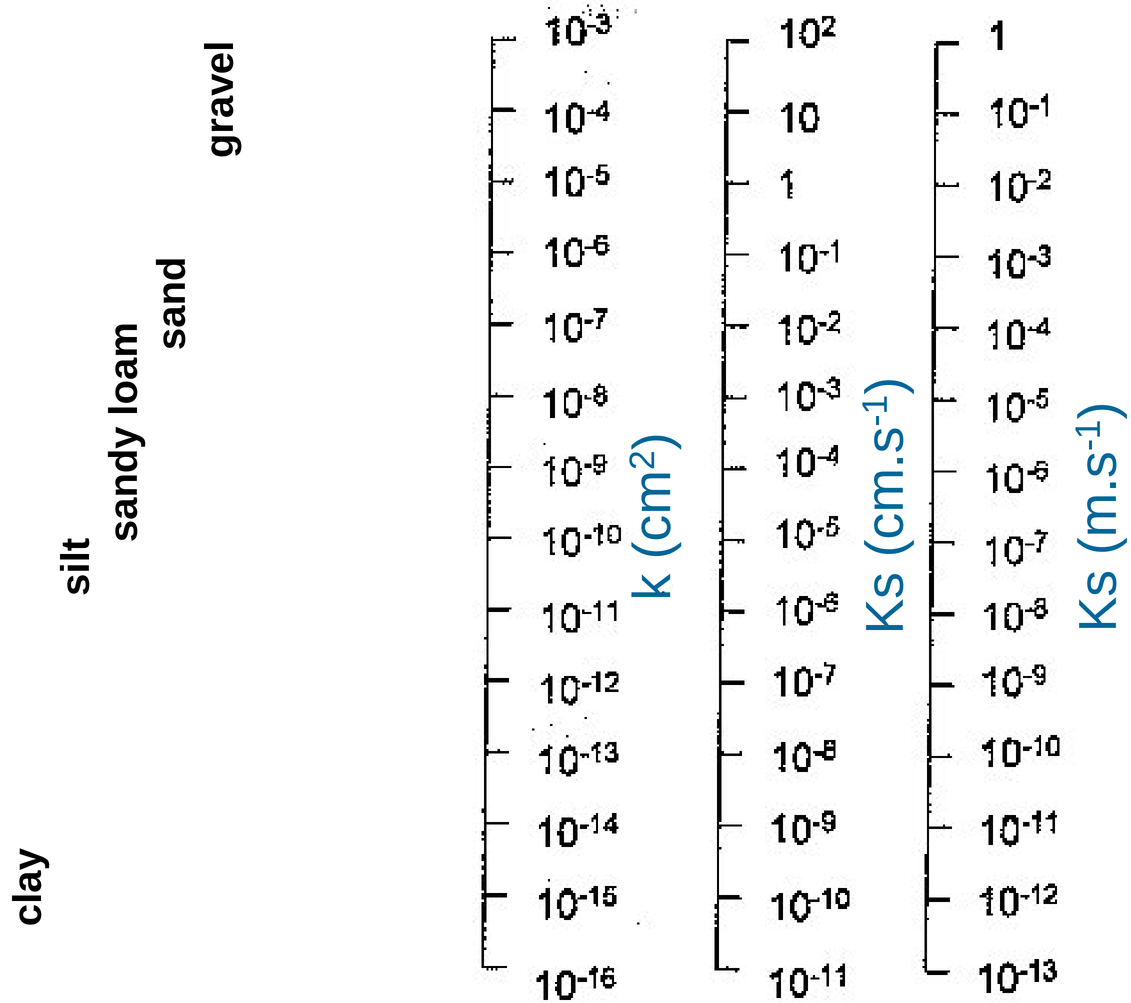
Porous velocity

$$v_p = \frac{Q}{S_n} = \frac{Q}{n\,S}$$

$$v_p = \frac{v}{n}$$

Where *n* is porosity

Ks for different media



source: Císlerová a Vogel, 1998

Permeability

Permeability reflects porous medium only - k [m^2].
Is used for transport of solutes in porous space

Permeability can be estimated from number of formulas by combination of porosity, grain distribution and shapes of particles

Empirical formulas: $$k = c\,d^2$$

where *d* is diameter of effective grain (d_{10}), c=45 for clayey sand, c=140 clear sand.

$$k = \frac{1}{\beta}\left[\frac{(1-n)^2}{n^3}\left(\frac{\alpha}{100}\sum_m \frac{P_m}{d_m}\right)^2\right]^{-1}$$

Where β=5, α is grain shape parameter, α=6 for spherical grains, α=7.7 for sharp grains, P_m is procents of grains found on the sieve with d_m diameter

Physically based equation Carman - Kozeny:

$$k = C_0 \frac{n^3}{(1-n)^2 M_S^2}$$

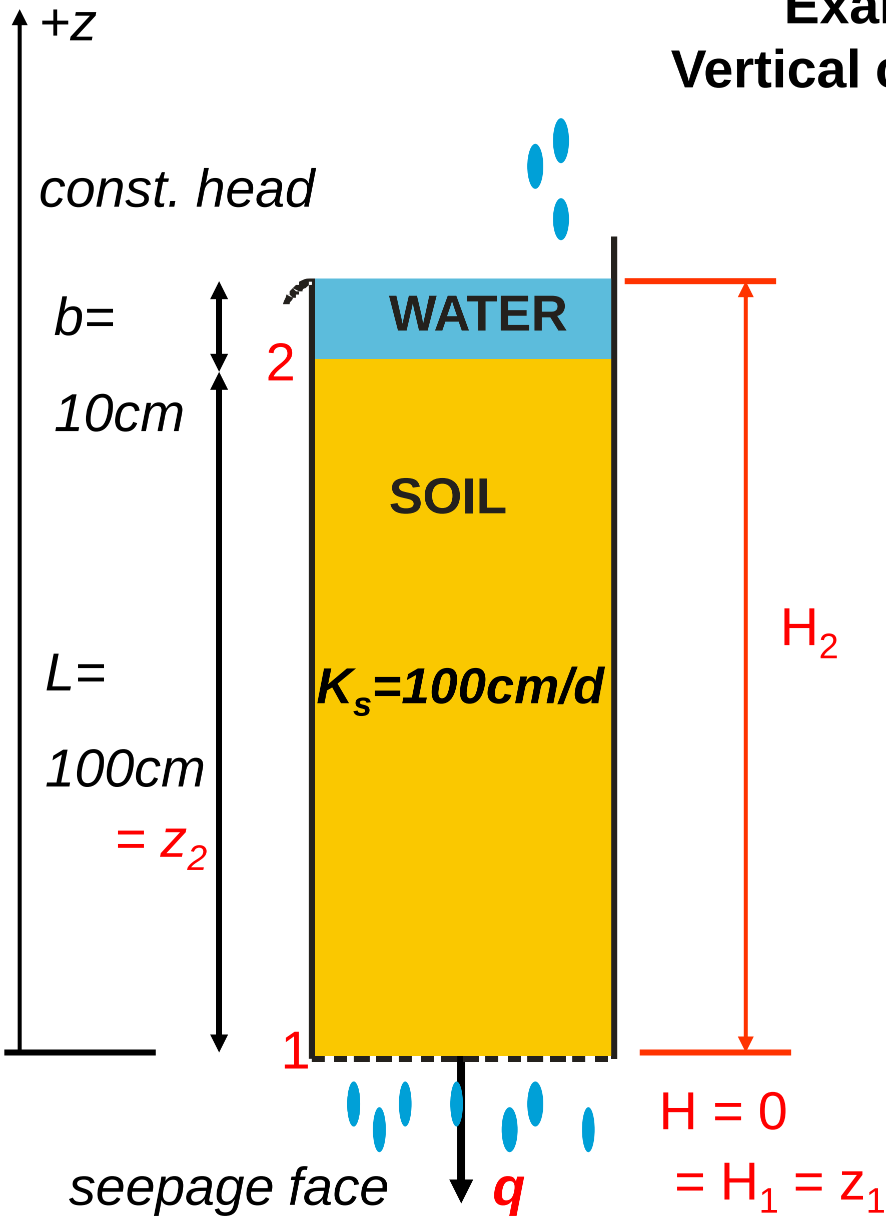
Where C_0 is empiric constant, M_S is specific surface of unit volume of porous space

Formula of Carman – Kozeny for hydraulic conductivity

$$K = \frac{g n^{3/2} e^2 d_e^2}{72 \nu}$$

Where n is porosity, e is number of porosity, d_e is effective grain
g is gravity acceleration, v is kinematic viscosity of water

Example 1 :
Vertical column: q=?



1) datum definition

2) points 1 and 2 with known hydraulic heads

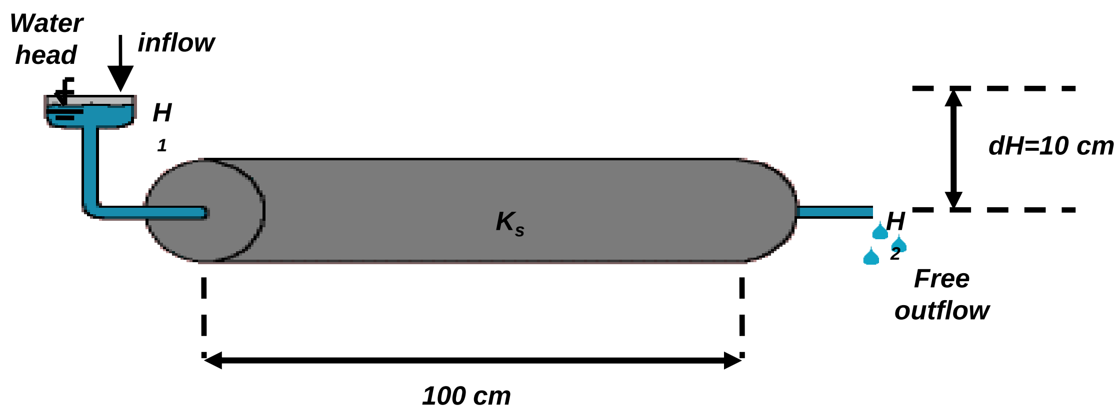
3) Darcy's law

$$q = -K_s \frac{\Delta H}{L} = -K_s \frac{(H_2 - H_1)}{L} =$$

$$= -100 \frac{(110 - 0)}{100} = -110 \; cm.d^{-1}$$

Example 2
horizontal column: q = ?

1) Step 1, definition of datum and coordination system



2) Definition of points 1 and 2). Then $x_1 = 0$ and $h_1 = 10$ cm, x_2=100cm, $h_2 = 0$, $z_1 = z_2 = 0$, $L = x_2 - x_1 = 100$ cm

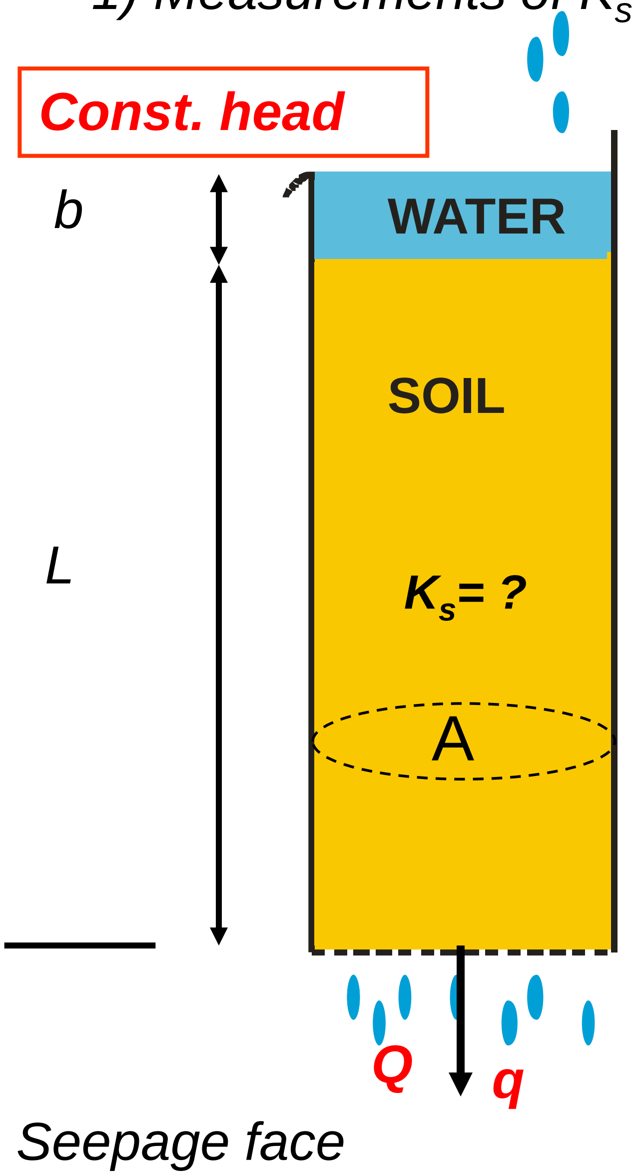
3) Hydraulics heads are then $H_1 = h_1 + z_1 = 10$ cm, $H_2 = h_2 + z_2 = 0$ cm

5) Darcy's law

$$q = -K_s \frac{\Delta H}{L} = -K_s \frac{(H_2 - H_1)}{L} = -100 \frac{(0-10)}{100} = 10 \; cm.d^{-1}$$

K_s measurements

1) Measurements of K_s using constant head permeameter



Soil column

$H_1 = 0 + 0$ *(lower end)*

$H_2 = b + L$ *(upper end)*

$\Delta H = (b+L) - 0$

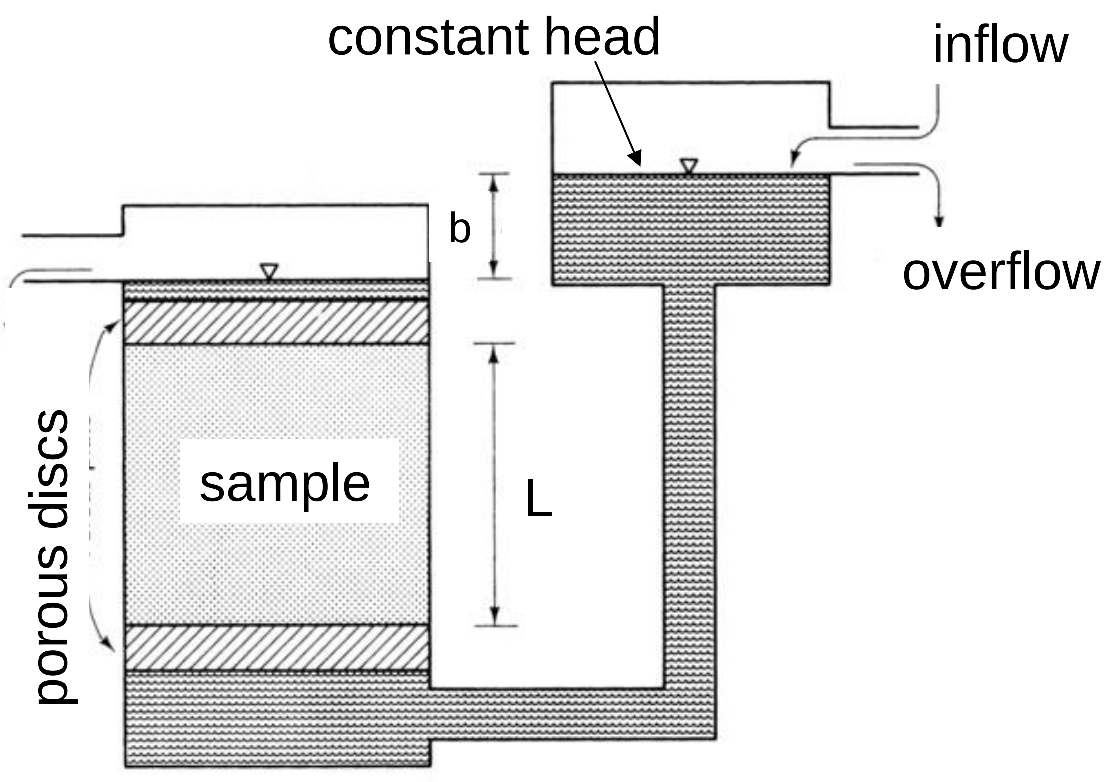
then:

$$K_s = -\frac{qL}{\Delta H} = -\frac{qL}{(b+L)}$$

In practice we measure Q, resp. V/t, then:

$$K_s = \frac{VL}{At\Delta H} = \frac{VL}{At(b+L)}$$

Constant head permeameter



Q

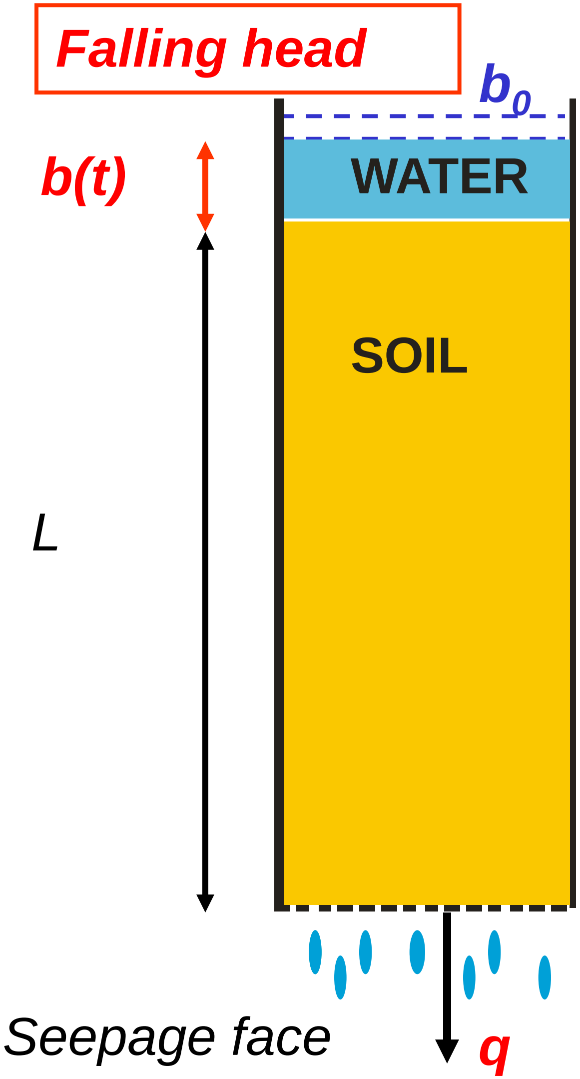
measurement

soil sample must be carefully saturated with water to measure “real” ***Ks****.*

Fig. : http://www.utdallas.edu/~brikowi/Teaching/Geohydrology/LectureNotes/Darcy_Law/Permeameters.html

Constant head permeameter

2) Measurement of K_s falling-head permeameter



Experiment is done on soil sample in the laboratory. Initial water level is equal to b_0

$H_1 = 0,\ H_2(t) = L + b(t), \quad \Delta H(t) = [b(t) + L] - 0$

$$q = \frac{db}{dt} = -K_s \frac{(b+L)}{L}$$

becomes:

$$\frac{db}{b+L} = -\frac{K_s}{L} dt$$

Integration of the left side

$$\int_{b_0}^{b_1} \frac{db}{b+L} = \ln(b+L)\Big|_{b_0}^{b_1} = \ln\frac{b_1+L}{b_0+L}$$

Right side integration

$$-\int_0^{t_1} \frac{K_s}{L} dt = -\frac{K_s}{L}\int_0^{t_1} dt = -\frac{K_s t_1}{L}$$

$$\frac{db}{b+L} = -\frac{K_s}{L} dt$$

then:

$$\ln\frac{b_1+L}{b_0+L} = -\frac{K_s t_1}{L}$$

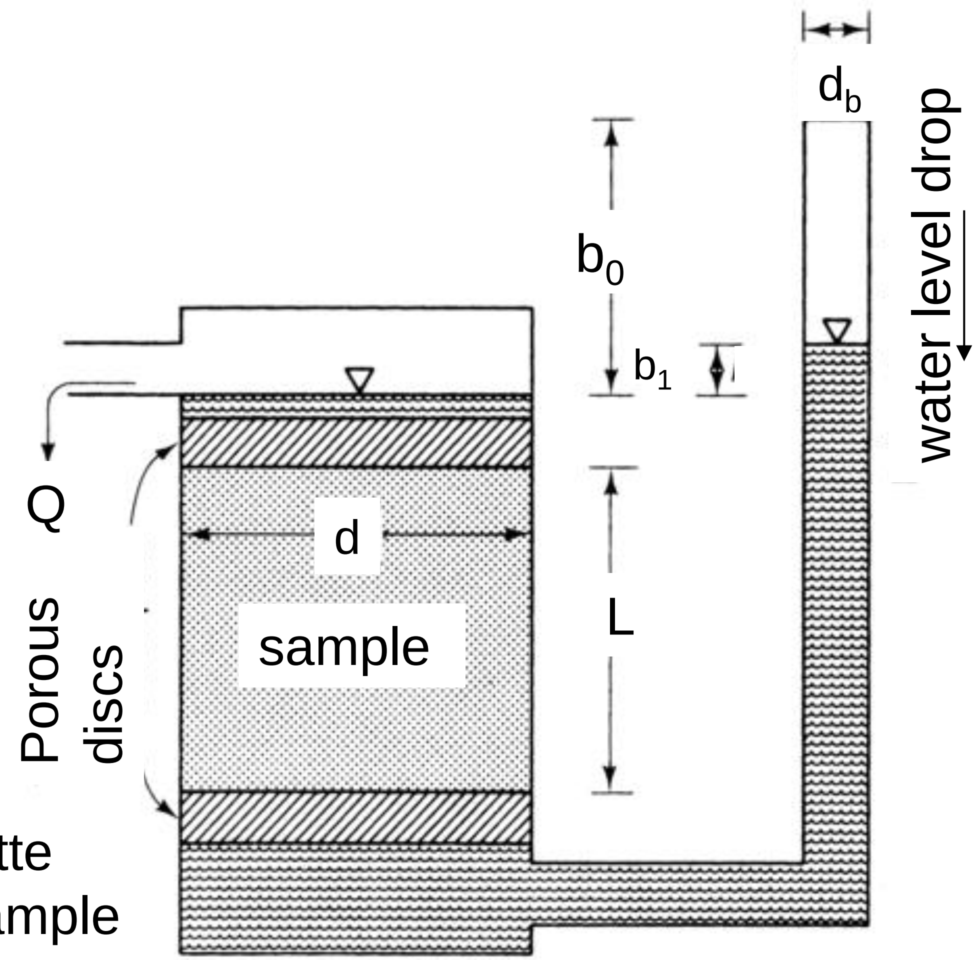
$$\boxed{K_s = \frac{L}{t_1}\ln\frac{b_0+L}{b_1+L}}$$

Falling-head permeameter

Consider different crossectional areas of burette and column:

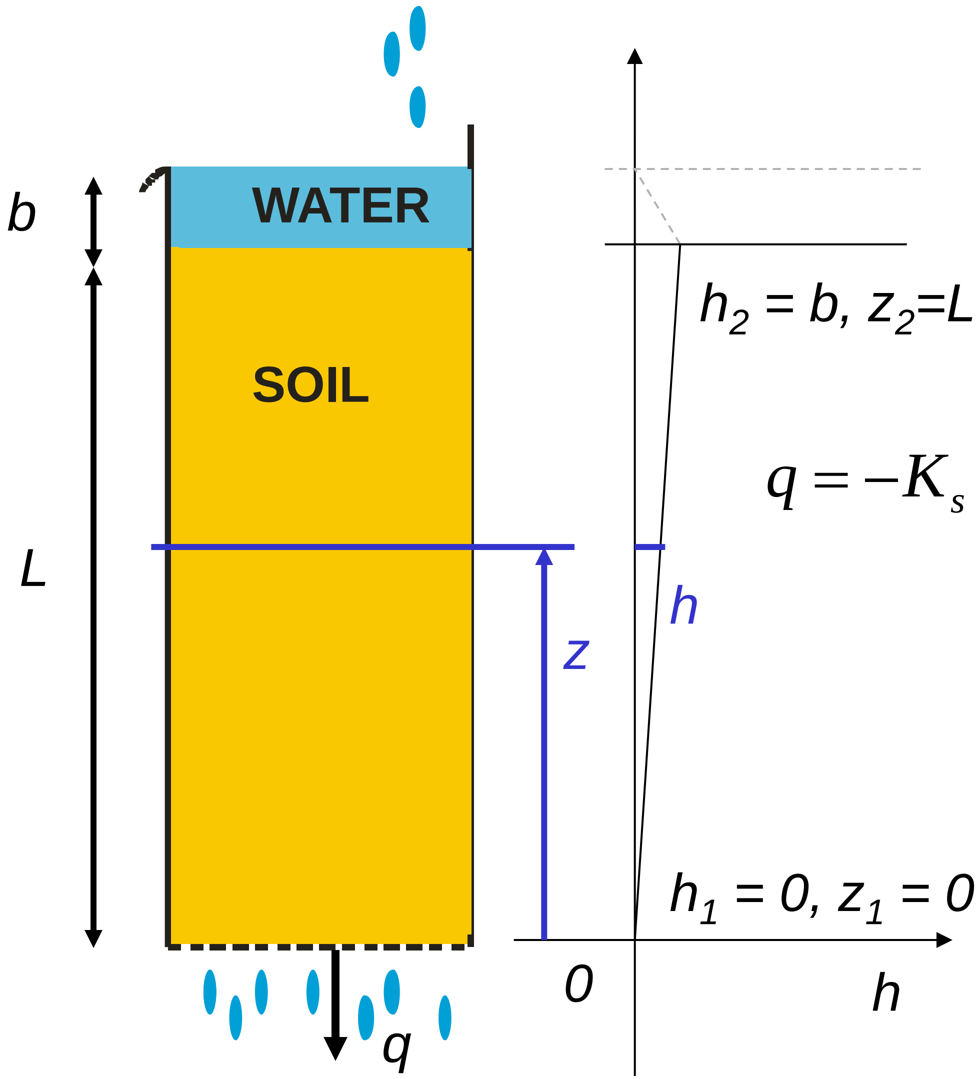
$$K_s = \frac{A_b}{A}\frac{L}{t_1}\ln\frac{b_0+L}{b_1+L}$$

where:
t_1.... time
A_b … cross. area of burette
A ... cross. area of the sample



Zdroj: http://www.utdallas.edu/~brikowi/Teaching/Geohydrology/LectureNotes/Darcy_Law/Permeameters.html

Example 3 :
Pressure head along the soil column h(z)=?



K_s is constant, $h(z)$ = ?

Darcy's law:

$$q = -K_s \frac{\Delta H}{L} = \underbrace{-K_s \frac{b+L}{L}}_{q_{12}} = \underbrace{-K_s \frac{h+z}{z}}_{q_{1z}}$$

$$h = \frac{h_2 - h_1}{L} z = \frac{b}{L} z$$

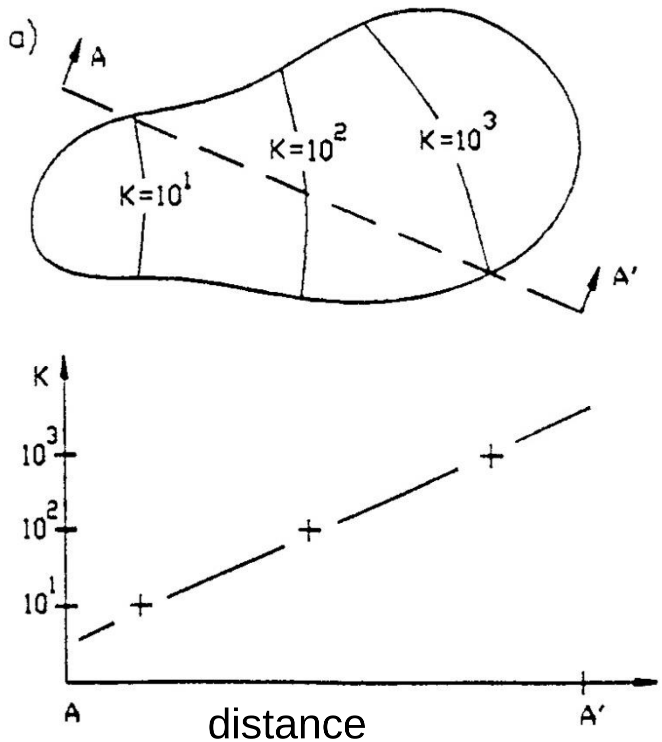
In this case, pressure head profile is linear .

Homogeneity and inhomogeneity of the environment with regards to hydraulic conductivity

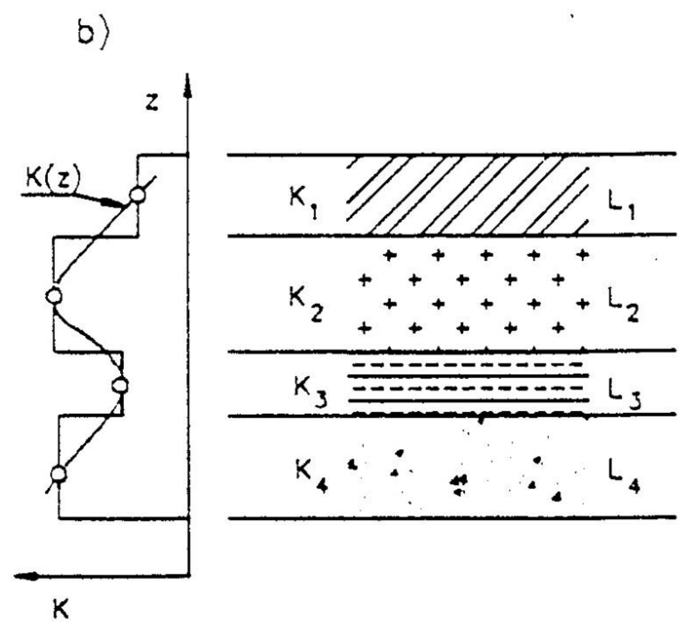
The environment is homogeneous: hydraulic conductivity is same at all points of the environment

The environment is inhomogeneous = heterogeneous: environment where hydraulic conductivity varies according to position in the area

Heterogeneity with gradual change

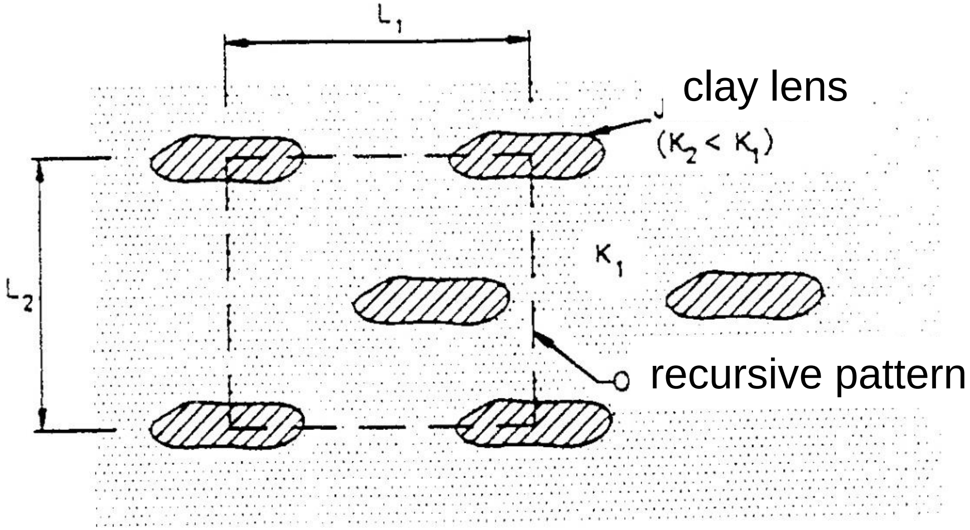


Heterogeneity with sudden change



Practical situations lead us to considerations whether accept or neglect inhomogeneity

Example of inhomogeneous environment, accepted as homogeneous



Calculating flow velocity

$$\mathbf{v} = -K\ \mathbf{J} = -K\ grad\ H \quad \text{or} \quad v_x = -K\frac{\partial H}{\partial x}, \quad v_y = -K\frac{\partial H}{\partial y}, \quad v_z = -K\frac{\partial H}{\partial z}$$

Where K is the function of x,y and z.

Isotropy and anisotropy of the environment

The environment is isotropic when hydraulic conductivity is same at all directions

The environment is anisotropic when hydraulic conductivity is direction dependent, e.g.
Hydraulic conductivity in vertical direction is lower than in horizontal direction
This property is due to the formation of the structure (e.g. sediments)

Hydraulic conductivity of anisotropic environment is defined by tensor of hydraulic conductivity

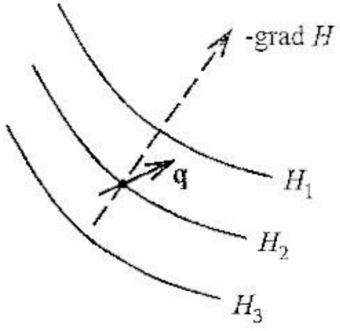
3D

$$\mathbf{K} = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}$$

2D

$$\mathbf{K} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix}$$

Component K_{xy} defines part of the velocity in x direction caused by unit hydraulic gradient in y direction



Vector of velocity is tilted from the direction of the hydraulic gradient in anisotropic environment

Sedimentary rock – horizontal structure

https://commons.wikimedia.org/w/index.php?curid=8912695

Eolic sediment – vertical structure

- https://commons.wikimedia.org/w/index.php?curid=7736690

Main axes of anisotropy are the direction where hydraulic conductivity reaches maximal or minimal value

If coordinate system x,y,z is parallel to main axes of anisotropy, tensor of hydraulic conductivity is

3D

$$\mathbf{K} = \begin{bmatrix} K_{xx} & 0 & 0 \\ 0 & K_{yy} & 0 \\ 0 & 0 & K_{zz} \end{bmatrix}$$

2D

$$\mathbf{K} = \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix}$$

Components of flow in anisotropic environemnt are following

$$v_x = -K_{xx}\frac{\partial H}{\partial x} - K_{xy}\frac{\partial H}{\partial y} - K_{xz}\frac{\partial H}{\partial z}$$

$$v_y = -K_{yx}\frac{\partial H}{\partial x} - K_{yy}\frac{\partial H}{\partial y} - K_{yz}\frac{\partial H}{\partial z}$$

$$v_z = -K_{xz}\frac{\partial H}{\partial x} - K_{yz}\frac{\partial H}{\partial y} - K_{zz}\frac{\partial H}{\partial z}$$

or

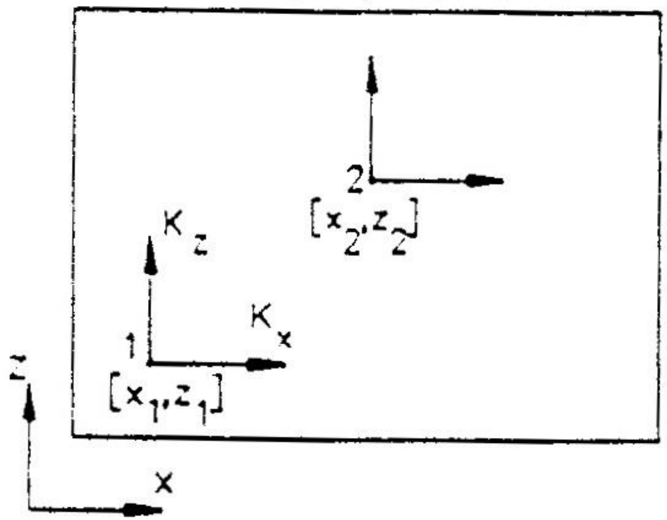
$$v_x = -K_x\frac{\partial H}{\partial x}$$

$$v_y = -K_y\frac{\partial H}{\partial y}$$

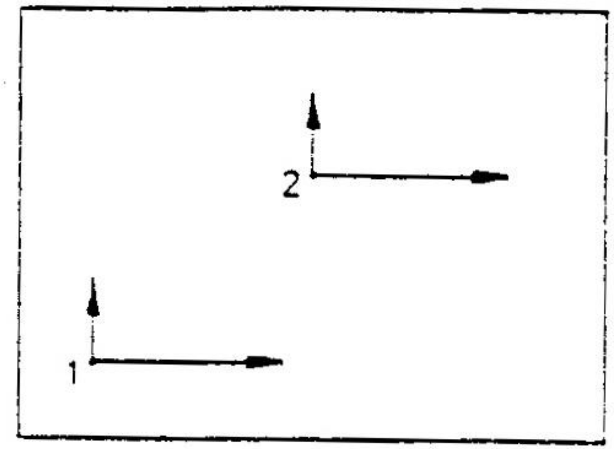
$$v_z = -K_z\frac{\partial H}{\partial z}$$

Combination of homogeneity (1) and isotropy (2) in the in environment

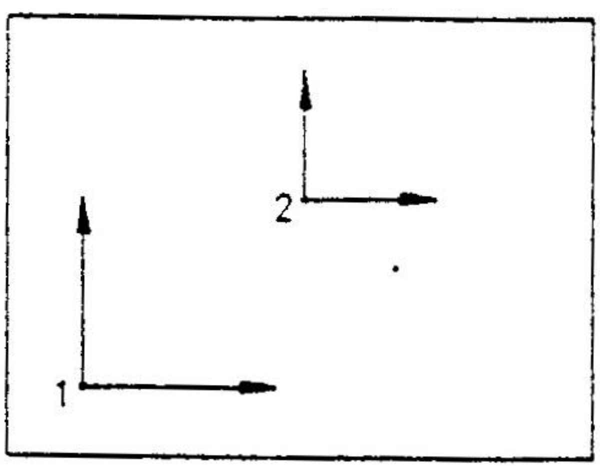
homogeneous, isotropic



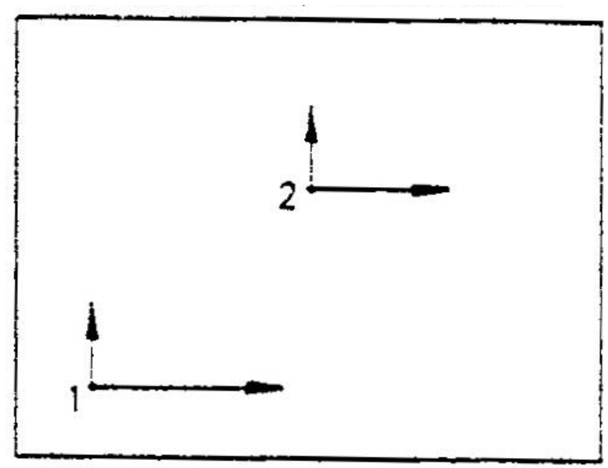
homogeneous, anisotropic



inhomogeneous, isotropic

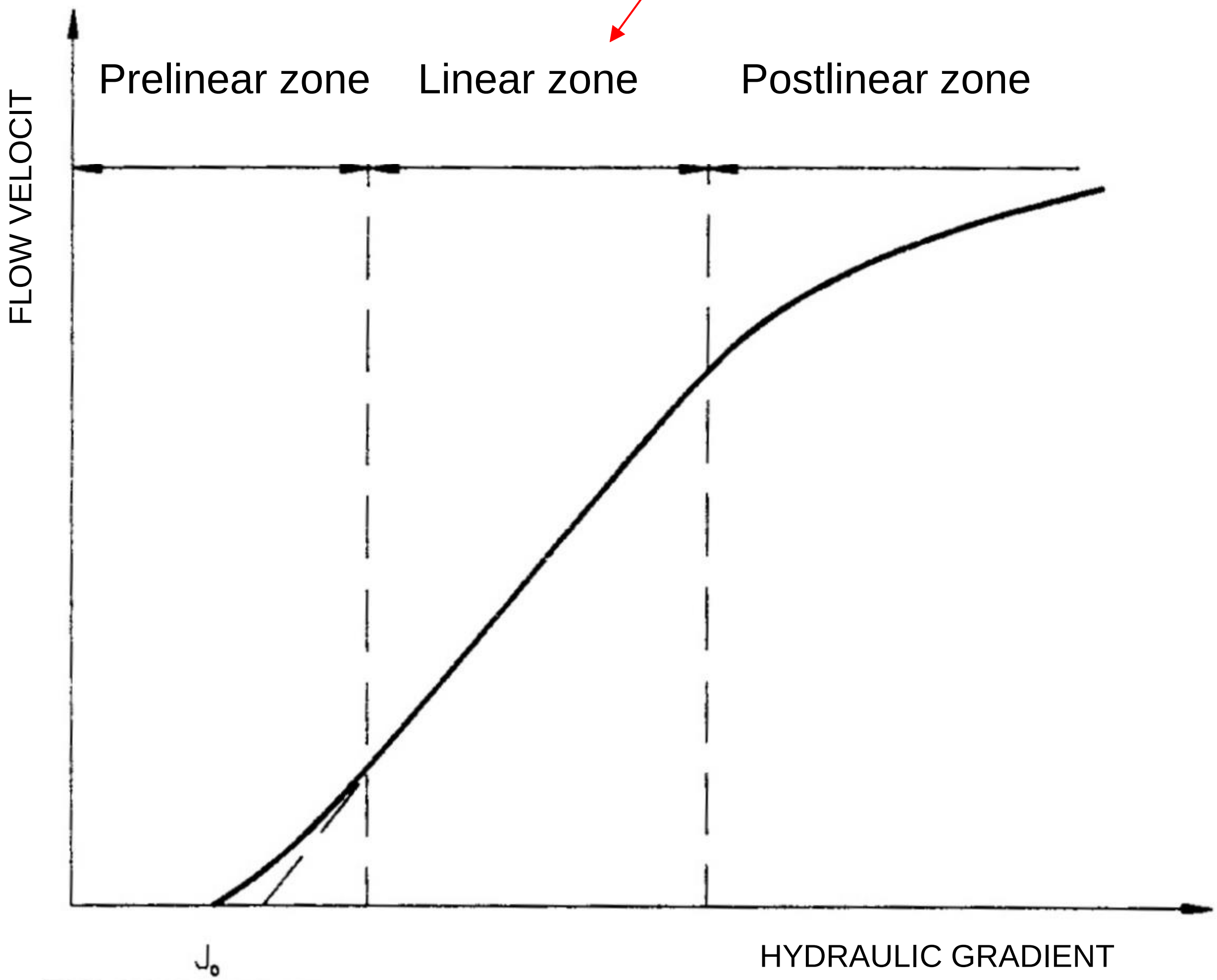


inhomogeneous, anisotropic

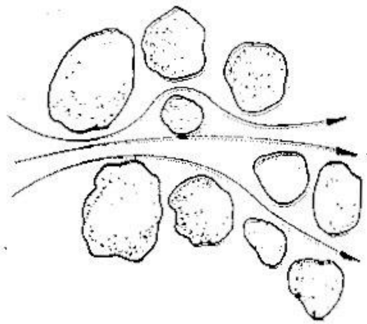


Limits of Darcy’s law validity

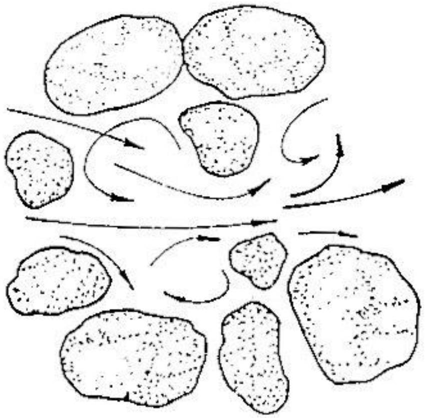
Darcy’s law is valid in certain interval of hydraulic gradient and flow velocities only



Laminar flow



Turbulent flow



Prelinear flow

Happens in very **fine materials** (clay and silt), flow velocity is calculated by Swartzendruber:

$$v = K\left(J - \frac{4}{3} J_0 + exp(\omega J)\right) \text{ where } \omega = \frac{1}{J_0} ln \frac{J_0}{3} \ .$$

J_0 is limit value of hydraulic gradient, where flow is actuvated.
J_0 < 0.5 for silt, J_0= 0.5 - 1.0 for clay.

Postlinear flow

Happens in **coarse materials** (gravel or sandy gravel), where higher velocities occur

Type of flow: laminar vs. turbulent is indicated by Reynolds number:

$$Re = \frac{\mathrm{v}\, d}{\nu}$$

Critical value of Re is dependent on given material. Typically 10 - 100 (1 - 10).
D is grain size and ν is kinematic viscosity

For postlienar zone, flow can be calculated by combined method

$$grad\ \phi = A|\mathrm{v}| + B|\mathrm{v}|^2 \quad \text{or} \quad grad\ \phi = A|\mathrm{v}| + B|\mathrm{v}|^m,\ 1.6 \le m \le 2$$

A, B are constants, characterizing permeability of the environment, d or de is effective grain:

$$A = 1/K \qquad B = 1/K_t^2, \qquad K = \frac{gn^{3/2}(ed)^2}{72\nu} \qquad \text{a} \qquad \mathrm{K_t} = 4\sqrt{gd}\ n^{5/4}\log_{10}(4.933e)$$

Pavlovskij equation

$$|\mathrm{v}| = K\,(grad\ \phi)^{0.5} \qquad\qquad K = n\left(20\sqrt{d_e} - \frac{14}{\sqrt{d_e}}\right)$$

In most of the cases groundwater flow meets linear zone where Darcy’s law is applicable. Upper limit of Darcy’s law might be overcome in carstic, dolomitic or volcanic areas with cavities.

Transmissivity

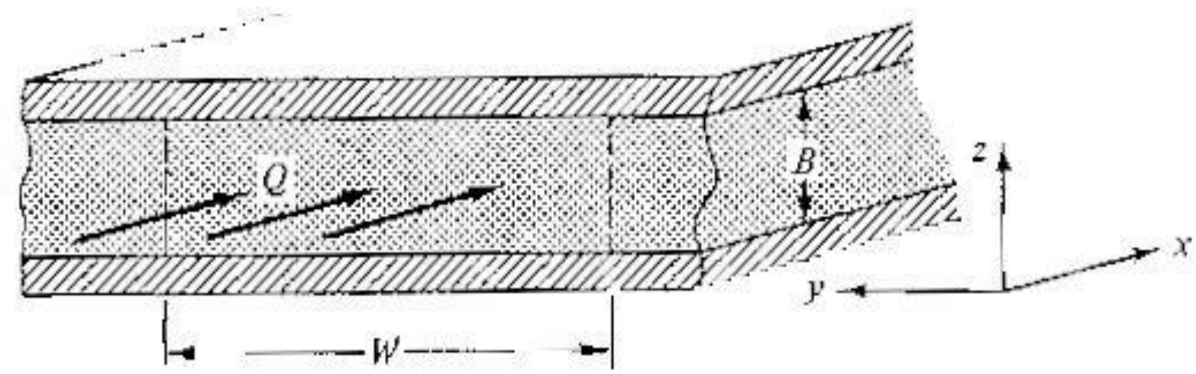
Transmissivity – is ability of aquifer to transfer water in one horizontal meter of aquifer of thickness B and hydraulic conductivity K

$$T = KB$$

Transmissivity units [m^2/s]

$$Q_x = -K\,B W \frac{\partial H}{\partial x}$$

$$Q_y = -K\,B W \frac{\partial H}{\partial y}$$



Transmissivity can be applied if vertical flow component is not considered